# On the Existence of Equilibria in Discontinuous Games: Three Counterexamples

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**Abstract**

We study whether we can weaken the conditions given in Reny [4] and still obtain existence of pure strategy Nash equilibria in quasiconcave normal form games, or, at least, existence of pure strategy $\varepsilon-$equilibria for all $\varepsilon > 0$. We show by examples that there are:

1. quasiconcave, payoff secure games without pure strategy $\varepsilon-$equilibria for small enough $\varepsilon > 0$ (and hence, without pure strategy Nash equilibria),
2. quasiconcave, reciprocally upper semicontinuous games without pure strategy $\varepsilon-$equilibria for small enough $\varepsilon > 0$, and
3. payoff secure games whose mixed extension is not payoff secure.

The last example, due to Sion and Wolfe [6], also shows that non-quasiconcave games that are payoff secure and reciprocally upper semicontinuous may fail to have mixed strategy equilibria.

# 1 Introduction

Recently, there has been an attempt to extent Nash's existence result [3] from finite normal form games to infinite-action games with discontinuous payoff functions (see Baye et al. [1], Dasgupta and Maskin [2], Reny [4] and Simon [5]). The most general result is due to Reny [4], which showed that a quasiconcave normal form game has a pure strategy Nash equilibrium if it is better-reply secure.

Better-reply security combines and generalizes two intuitive conditions, payoff security[1] and reciprocal upper semicontinuity.[2] Our goal is to study whether any such condition alone is enough to guarantee, in quasiconcave games, the existence of pure strategy Nash equilibria or, at least, the existence of pure strategy $\varepsilon-$equilibria, for all $\varepsilon > 0$.

We provide two examples that show that we cannot drop either of the two condition and still obtain a pure strategy Nash equilibrium. Both games in the examples are simples games, games in which each player's payoff function is simple (i.e., has a finite range). Since a simple game has a Nash equilibrium if and only if it has $\varepsilon-$equilibria for all $\varepsilon > 0$ (see proposition 2), our examples also show that neither payoff security nor reciprocal upper semicontinuity alone are enough to guarantee the existence of pure strategy $\varepsilon-$equilibrium, for all $\varepsilon > 0$, in quasiconcave games.

When applied to the mixed extension of a normal form game, Reny's Theorem asserts that a mixed strategy equilibrium exists for a given normal form game provided that its mixed extension is payoff secure and reciprocally upper semicontinuous. It is useful to know when we can conclude that the mixed extension of a given normal form game satisfies those conditions by studying the properties of the original game, since the analysis of the latter is typically easier than that of its mixed extension. In this line of research, Reny showed that a sufficient condition for the mixed extension of a given normal form game to be reciprocally upper semicontinuous is that the sum of the payoff function of the original game is upper semicontinuous.

Given the above result, we ask whether the payoff security of the mixed extension of a given normal form game follows from the payoff security of

[1] "Payoff security requires that for every strategy $x$, each player has a strategy that virtually guarantees the payoff he receives at $x$, even if the others deviate slightly from $x$" (Reny [4]).

[2] "Reciprocal upper semicontinuity requires that some player's payoff jumps up whenever some other player's payoff jumps down" (Reny [4]).

the original game. We use the example in Sion and Wolfe [6] to show that this conjecture is false. This implies that non-quasiconcave, payoff secure, reciprocally upper semicontinuous games may fail to have mixed strategy equilibria.

## 2 Normal form games

A *normal form game* $G$ consists of a finite set of players $N = \{1, ..., n\}$, and, for every player $i \in N$, a pure strategy set $X_i$, represented by a nonempty topological space, and a payoff function $U_i : X \to \mathbb{R}$, where $X = \times_{i \in N} X_i$. A normal form game is said to be *simple* if for all $i \in N$, $U_i$ is a simple function (i.e., a function with the property that its range is a finite set).

Throughout, the product of any number of sets is endowed with the product topology. Given a player $i \in N$, the symbol $-i$ denotes "all players but $i$." In particular, $X_{-i} = \times_{j \neq i} X_j$.

The vector of the players' payoff functions will be denoted by $U : X \to \mathbb{R}^N$ and is defined by $U(x) = (U_1(x), ..., U_n(x))$ for every $x \in X$. The graph of $U$ is the subset of $X \times \mathbb{R}^n$ given by $\{(x, u) \in X \times \mathbb{R}^n : u = U(x)\}$. It will be denoted by graph$(U)$.

Given a player $i \in N$, we say that *player $i$ can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$* if there exists $\tilde{x}_i \in X_i$ and a neighborhood $V_{x_{-i}}$ of $x_{-i}$ such that $U_i(\tilde{x}_i, x'_{-i}) \geq \alpha$ for all $x'_{-i} \in V_{x_{-i}}$ (see Reny [4]).

A payoff function $U_i$ for player $i \in N$ is *payoff secure* if for every $x \in X$ and every $\varepsilon > 0$, player $i$ can secure a payoff of $U_i(x) - \varepsilon$ at $x$ (see Reny [4]). A normal-form game $G = \langle N, (X_i, U_i)_{i \in N} \rangle$ is payoff secure if for all $i \in N$, $U_i$ is payoff secure.

A game $G = \langle N, (X_i, U_i)_{i \in N} \rangle$ is *reciprocally upper semicontinuous* if for all $(x, u) \in \overline{\text{graph}(U)}$ such that $U_i(x) \leq u_i$ holds for all $i \in N$, then $U_i(x) = u_i$ for all $i$ (see Simon [5] and Reny [4]).[3]

A game $G = \langle N, (X_i, U_i)_{i \in N} \rangle$ is *quasiconcave* if for all $i \in N$, $X_i$ is convex and $U_i(\cdot, x_{-i})$ is quasiconcave on $X_i$, for all $x_{-i} \in X_{-i}$.

Let $M_i$ be the set of all regular probability measures in $X_i$ and $M = \times_{i \in N} M_i$. If $U_i$ is Borel measurable and integrable with respect to all $\mu \in M_i$, we define $v_i : M \to \mathbb{R}$ by

$$v_i(\mu) = \int_X U_i d\mu. \tag{1}$$

[3]For any subset $A$ of a topological space $Y$, $\overline{A}$ denotes the closure of $A$.

Finally, we define the mixed extension of $G$ to be $\tilde{G} = \langle N, (M_i, v_i)_{i \in N} \rangle$.

Given a normal form game $G = \langle N, (X_i, U_i)_{i \in N} \rangle$, a *pure strategy Nash equilibrium of* $G$ is $x^* \in X$ such that, for all $i \in N$ and $x_i \in X_i$,

$$U_i(x^*) \geq U_i(x_i, x^*_{-i}). \tag{2}$$

Given $\varepsilon > 0$, a *pure strategy* $\varepsilon-$*equilibrium of* $G$ is $x^* \in X$ such that, for all $i \in N$ and $x_i \in X_i$,

$$U_i(x^*) \geq U_i(x_i, x^*_{-i}) - \varepsilon. \tag{3}$$

A *mixed strategy Nash equilibrium of* $G$ is a pure strategy Nash equilibrium of $\tilde{G}$, and a *mixed strategy* $\varepsilon-$*equilibrium of* $G$ is a pure strategy $\varepsilon-$equilibrium of $\tilde{G}$.

# 3 On the Existence of Nash Equilibria

The following is an example of a simple, quasiconcave and payoff secure game that doesn't possess any pure strategy Nash equilibrium. Since a simple game has a pure strategy Nash equilibrium if and only if it has an $\varepsilon-$equilibrium for all $\varepsilon > 0$ (proposition 2), we conclude that payoff security is not enough to guarantee the existence of pure strategy $\varepsilon-$equilibrium, for all $\varepsilon > 0$, in quasiconcave games.

**Example 1** *Let* $G_1$ *be described by* $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, $U_1 : X \to \mathbb{R}$ *be defined by*

$$U_1(x_1, x_2) = \begin{cases} 0 & \textit{if } x_2 \leq \frac{1}{2} - x_1; \\ 2 & \textit{if } x_1 = 0 \textit{ and } x_2 > \frac{1}{2}; \\ 1 & \textit{otherwise}, \end{cases} \tag{4}$$

*and* $U_2 : X \to \mathbb{R}$ *be defined by*

$$U_2(x_1, x_2) = \begin{cases} 0 & \textit{if } x_1 \leq \frac{1}{2} \textit{ and } x_2 > 0; \\ 1 & \textit{if } x_1 \leq \frac{1}{2} \textit{ and } x_2 = 0; \\ 1 & \textit{if } x_1 > \frac{1}{2} \textit{ and } x_2 \leq \frac{1}{2}; \\ 2 & \textit{if } x_1 > \frac{1}{2} \textit{ and } x_2 > \frac{1}{2}. \end{cases} \tag{5}$$

Their graphs are illustrated in figure 1:

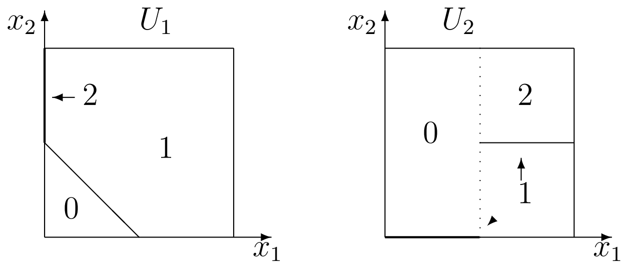

Figure 1

**Proposition 1** *The game $G_1$ is quasiconcave and payoff secure but has no pure strategy $\varepsilon-$equilibrium, if $\varepsilon > 0$ is small enough. In particular, $G_1$ has no pure strategy Nash equilibrium.*

**Proof.** One easily checks that $G_1$ is quasiconcave: Let $\alpha \in \mathbb{R}$. For $0 \leq x_2 \leq 1/2$,

$$\{x_1 : U_1(x_1, x_2) \geq \alpha\} = \begin{cases} [0,1] & \text{if } \alpha \leq 0, \\ (\frac{1}{2} - x_2, 1] & \text{if } 0 < \alpha \leq 1, \\ \emptyset & \text{otherwise,} \end{cases} \tag{6}$$

and for $1/2 < x_2 \leq 1$,

$$\{x_1 : U_1(x_1, x_2) \geq \alpha\} = \begin{cases} [0,1] & \text{if } \alpha \leq 1, \\ \{0\} & \text{if } 1 < \alpha \leq 2, \\ \emptyset & \text{otherwise.} \end{cases} \tag{7}$$

Similarly, for $0 \leq x_1 \leq 1/2$,

$$\{x_2 : U_2(x_1, x_2) \geq \alpha\} = \begin{cases} [0,1] & \text{if } \alpha \leq 0, \\ \{0\} & \text{if } 0 < \alpha \leq 1, \\ \emptyset & \text{otherwise,} \end{cases} \tag{8}$$

and for $1/2 < x_1 \leq 1$,

$$\{x_2 : U_2(x_1, x_2) \geq \alpha\} = \begin{cases} [0,1] & \text{if } \alpha \leq 1, \\ (\frac{1}{2}, 1] & \text{if } 1 < \alpha \leq 2, \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

Hence, $G_1$ is quasiconcave.

As for payoff security: to show that $U_1$ is payoff secure it is enough to show that $U_1$ is payoff secure at $(x_1, x_2)$ satisfying $x_1 = 0$ and $x_2 > 1/2$. This is so because in points $x$ at which $U_1(x) = 0$ there is noting to show, and in the remaining case, i.e., when $U_1(x) = 1$, $U_1$ is continuous. If $x_1 = 0$ and $x_2 > 1/2$ then $U_1(x_1, x_2) = 2$. Let $\tilde{x}_1 = 0$ and $V_{x_2}$ be the open ball centered at $x_2$ with radius $\delta = x_2 - 1/2$. Then $U_1(\tilde{x}_1, x_2') = 2$ for all $x_2' \in V_{x_2}$. Similarly for $U_2$ it is enough to consider the following two cases: the first is when $x_2 = 0$ and $0 \leq x_1 \leq 1/2$, in which case we let $\tilde{x}_2 = 0$; and the second is when $x_2 = 1/2$ and $x_1 > 1/2$, in which case we let $\tilde{x}_2 = 1/2$ and $V_{x_1}$ be the open ball centered at $x_1$ with radius $\delta = x_1 - 1/2$. Hence, $G_1$ is payoff secure.

Letting $\beta_i$ denote player $i$'s best reply correspondence, we obtain

$$\beta_1(x_2) = \begin{cases} (\frac{1}{2} - x_2, 1] & \text{if } x_2 \leq \frac{1}{2} \\ \{0\} & \text{if } x_2 > \frac{1}{2}. \end{cases} \tag{10}$$

and

$$\beta_2(x_1) = \begin{cases} \{0\} & \text{if } x_1 \leq \frac{1}{2}; \\ (\frac{1}{2}, 1] & \text{if } x_1 > \frac{1}{2}. \end{cases} \tag{11}$$

Since their graphs don't intercept, it follows that there is no pure strategy Nash equilibrium. Also, we can conclude from proposition 2 that $G$ has no pure strategy $\varepsilon-$equilibrium, if $\varepsilon > 0$ is small enough. ■

As claimed above, for simples games the existence of Nash equilibria is equivalent to the existence of $\varepsilon-$equilibria for all $\varepsilon > 0$.

**Proposition 2** *Let $G = \langle N, (X_i, U_i)_{i \in N} \rangle$ be a simple game. Then, $G$ has a pure strategy Nash equilibrium if and only if $G$ has a pure strategy $\varepsilon-$equilibrium, for all $\varepsilon > 0$.*

**Proof.** Since necessity is obvious, we prove only sufficiency.

For all $i \in N$, let $U_i(X) = \{d_1^i, \ldots, d_{L_i}^i\}$, with $d_1^i < \ldots < d_{L_i}^i$ and let $\varepsilon > 0$ be such that $\varepsilon < \min_{i \in N} \min_{l \in \{1, \ldots, L_i - 1\}} (d_{l+1}^i - d_l^i)$. It follows that if $x^* \in X$ is an $\varepsilon-$equilibrium, which exists by assumption, then $x^*$ is a Nash equilibrium, since $U_i(x^*) + \varepsilon \geq U_i(x_i, x_{-i}^*)$, for all $x_i \in X_i$ implies $U_i(x^*) \geq U_i(x_i, x_{-i}^*)$, for all $x_i \in X_i$, given the way $\varepsilon$ was constructed. ■

The following is an example of a simple, quasiconcave and reciprocally upper semicontinuous game that doesn't possess any pure strategy Nash equilibrium. Hence, we conclude that reciprocally upper semicontinuity is not enough to guarantee the existence of pure strategy $\varepsilon-$equilibrium in quasiconcave games.

**Example 2** *Let $G_2$ be described by $N = \{1,2\}$, $X_1 = X_2 = [0,1]$, $U_1 : X \to \mathbb{R}$ be defined by*

$$U_1(x_1,x_2) = \begin{cases} 2 & \textit{if } x_2 = 1 - x_1; \\ 1 & \textit{if } 0 < x_1 \leq \frac{2}{3} \textit{ and } x_2 = 1; \\ 0 & \textit{otherwise,} \end{cases} \tag{12}$$

*and $U_2 : X \to \mathbb{R}$ be defined by*

$$U_2(x_1,x_2) = \begin{cases} 2 & \textit{if } x_1 \leq \frac{1}{3} \textit{ and } x_2 = x_1; \\ 2 & \textit{if } x_1 \geq \frac{2}{3} \textit{ and } x_2 = x_1; \\ 1 & \textit{if } \frac{1}{3} < x_1 < \frac{2}{3} \textit{ and } x_2 = \frac{1}{3} + x_1; \\ 0 & \textit{otherwise.} \end{cases} \tag{13}$$

Their graphs are illustrated in figure 2:

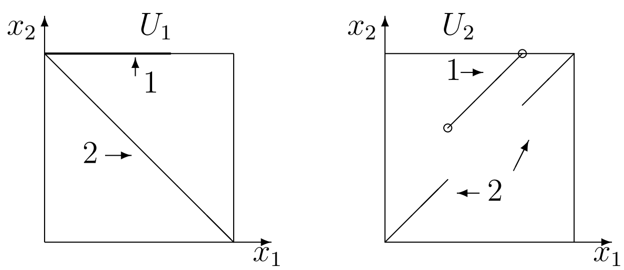


Figure 2

**Proposition 3** *The game $G_2$ is quasiconcave and reciprocally upper semicontinuous but has no pure strategy $\varepsilon-$equilibrium, if $\varepsilon > 0$ is small enough. In particular, $G_2$ has no pure strategy Nash equilibrium.*

**Proof.** We first show that $G_2$ is reciprocally upper semicontinuous: Let $\alpha \in \mathbb{R}$. We have that

$$\{U_1 + U_2 \geq \alpha\} = \begin{cases} \emptyset & \text{if } \alpha > 2, \\ \{U_1 + U_2 \geq 2\} & \text{if } 1 < \alpha \leq 2, \\ \{U_1 + U_2 \geq 1\} & \text{if } 0 < \alpha \leq 1, \\ X & \text{otherwise.} \end{cases} \tag{14}$$

Since

$$\begin{aligned} \{U_1 + U_2 \geq 2\} = &\{(x_1,x_2) : x_1 + x_2 = 1\} \cup \\ &\{(x_1,x_2) : x_1 = x_2 \text{ and } x_1 \leq 1/3\} \cup \\ &\{(x_1,x_2) : x_1 = x_2 \text{ and } x_1 \geq 2/3\}, \end{aligned} \tag{15}$$

and

$$
\begin{aligned}
\{U_1+U_2 \geq 1\} = &\{U_1+U_2 \geq 2\} \cup \{(x_1,x_2) : x_1 \leq 2/3 \text{ and } x_2 = 1\} \cup \\
&\{(x_1,x_2) : x_2 = 1/3 + x_1 \text{ and } 1/3 \leq x_1 \leq 2/3\},
\end{aligned} \tag{16}
$$

it follows that both $\{U_1+U_2 \geq 2\}$ and $\{U_1+U_2 \geq 1\}$ are closed. Hence, $U_1+U_2$ is upper semicontinuous and $G_2$ is reciprocally upper semicontinuous.

We show in what follows that $G_2$ is quasiconcave: Let $\alpha \in \mathbb{R}$. If $\alpha \leq 0$ then $\{x_1 : U_1(x_1,x_2) \geq \alpha\} = [0,1]$ and if $\alpha > 2$ then $\{x_1 : U_1(x_1,x_2) \geq \alpha\} = \emptyset$. If $0 < \alpha \leq 1$ then

$$
\{x_1 : U_1(x_1,x_2) \geq \alpha\} = \begin{cases} [0,\frac{2}{3}] & \text{if } x_2 = 1, \\ \{1-x_2\} & \text{otherwise,} \end{cases} \tag{17}
$$

and if $1 < \alpha \leq 2$,

$$
\{x_1 : U_1(x_1,x_2) \geq \alpha\} = \{1-x_2\}. \tag{18}
$$

Similarly, for $0 < \alpha \leq 1$,

$$
\{x_2 : U_2(x_1,x_2) \geq \alpha\} = \begin{cases} \{x_1\} & \text{if } 0 \leq x_1 \leq \frac{1}{3}, \\ \{\frac{1}{3}+x_1\} & \text{if } \frac{1}{3} < x_1 < \frac{2}{3}, \\ \{x_1\} & \text{otherwise,} \end{cases} \tag{19}
$$

and for $1 < \alpha \leq 2$,

$$
\{x_2 : U_2(x_1,x_2) \geq \alpha\} = \begin{cases} \{x_1\} & \text{if } 0 \leq x_1 \leq \frac{1}{3}, \\ \emptyset & \text{if } \frac{1}{3} < x_1 < \frac{2}{3}, \\ \{x_1\} & \text{otherwise.} \end{cases} \tag{20}
$$

Hence, $G_2$ is quasiconcave.

Letting $\beta_i$ denote player $i$'s best reply correspondence, we obtain

$$
\beta_1(x_2) = \{1-x_2\}, \tag{21}
$$

for all $x_2 \in X_2$ and

$$
\beta_2(x_1) = \begin{cases} \{\frac{1}{3}+x_1\} & \text{if } \frac{1}{3} < x_1 < \frac{2}{3}; \\ \{x_1\} & \text{otherwise.} \end{cases} \tag{22}
$$

Since their graphs don't intercept, it follows that there is no pure strategy Nash equilibrium. Hence, $G$ has no pure strategy $\varepsilon-$equilibrium, if $\varepsilon > 0$ is small enough. ■

When applied to the mixed extension of a normal form game $G$, Reny's Theorem 3.1 [4] asserts that a mixed strategy equilibrium exists for $G$ provided that $\tilde{G}$ is payoff secure and reciprocally upper semicontinuous. It is useful to know when we can conclude that $\tilde{G}$ satisfies those conditions by studying the properties of $G$, since the analysis of $G$ is typically easier than that of $\tilde{G}$. In this line of research, Reny showed that a sufficient condition for $\tilde{G}$ to be reciprocally upper semicontinuous is that $\Sigma_{i \in N} U_i$ is upper semicontinuous, because this will imply that $\Sigma_{i \in N} v_i$ is upper semicontinuous, which in turn implies that $\tilde{G}$ is reciprocally upper semicontinuous.

Given the above result, we ask whether the payoff security of $\tilde{G}$ follows from the payoff security of $G$. The following example shows that the fact that a game is payoff secure in pure strategies does not imply that it is payoff secure in mixed strategies.

**Example 3** *(Sion and Wolfe [6]) Let $G_3$ be described by $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, $U_1 : X \to \mathbb{R}$ be defined by*

$$U_1(x_1, x_2) = \begin{cases} -1 & \text{if } x_1 < x_2 < x_1 + \frac{1}{2}; \\ 0 & \text{if } x_1 = x_2 \text{ or } x_2 = x_1 + \frac{1}{2}; \\ 1 & \text{otherwise,} \end{cases} \tag{23}$$

*and $U_2 : X \to \mathbb{R}$ be defined by $U_2(x_1, x_2) = -U_1(x_1, x_2)$, for all $(x_1, x_2) \in X$.*

The graph of $U_1$ is illustrated in figure 3:

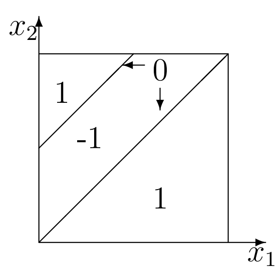


Figure 3

**Proposition 4** *The game $G_3$ is payoff secure but its mixed extension $\tilde{G}_3$ is not.*

**Proof.** One easily checks that $G_3$ is payoff secure. For player 1, it is enough to consider $x \in X$ such that $U_1(x) = 0$ since when $U_1(x) = -1$ there is noting to show, and $U_1$ is continuous in the remaining case (i.e., for $x$ such that $U_1(x) = 1$). If $x_1 = x_2$, we let $\tilde{x}_1 = 1$ except when $x_1 = 1$, in which case we let $\tilde{x}_1 = 0$; if $x_2 = x_1 + 1/2$, we let $\tilde{x}_1 = 0$ except when $x_1 = 1/2$, in which case we let $\tilde{x}_1 = 1$. In all these cases we can find a neighborhood $V_{x_2}$ of $x_2$ such that $U_1(\tilde{x}_1, x_2') = 1$ for all $x_2' \in V_{x_2}$.

For player 2, it is again enough to consider $x \in X$ such that $U_1(x) = 0$. We let $\tilde{x}_2 = x_1 + 1/4$ except when $x = (x_1, x_2) = (1, 1)$, in which case we let $\tilde{x}_2 = 1$. In all these cases we can find a neighborhood $V_{x_1}$ of $x_1$ such that $U_2(x_1', \tilde{x}_2) = 1$ for all $x_1' \in V_{x_1}$.

Also, since $G_3$ is a zero-sum game, so is $\tilde{G}_3$; hence, $\tilde{G}_3$ is reciprocally upper semicontinuous. However, as Sion and Wolfe [6] have shown, $G_3$ has no Nash equilibrium (pure or mixed), and so it follows from Reny's Theorem that $\tilde{G}_3$ is not payoff secure. ■

Sion and Wolfe [6] have shown that $G_3$ has no Nash equilibrium (pure or mixed); hence, $G_3$ is also an example of a (non-quasiconcave) reciprocally upper semicontinuous and payoff secure game without Nash equilibria.

# References


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